WESLEY COLLEGE

By daring & by doing

YEAR 12 MATHEMATICS SPECIALIST
SEMESTER ONE 2018
TEST 3: Vectors and Vector Calculus

Name: Solutions.

Friday 11 May

Mark /48 = %

- Answer all questions neatly in the spaces provided. **Show all working.**
- You are permitted to use the Formula Sheet in **both** sections of the test.
- You are permitted one A4 page (one side) of notes in the calculator assumed section.

Calculator free section **Suggested time: ~25 minutes** **/22**

Question 1 (3 marks)

The vector form of a sphere is given by $\left|\underset{\sim}{r} - \left(3\underset{\sim}{i} - 2\underset{\sim}{j} + 4\underset{\sim}{k}\right)\right| = 6$.

a) Give the Cartesian equation of the sphere.

$$(x-3)^2 + (y+2)^2 + (z-4)^2 = 36$$

[1]

b) Does the point $(2, 1, -5)$ lie inside, outside or on the sphere? Justify your answer.

$$\sqrt{[(2-3)^2 + (1+2)^2 + (-5-4)^2]} = \sqrt{[1+9+81]} = \sqrt{91} \checkmark$$

$\Rightarrow$ outside the sphere. ✓

[2]

Question 2 (3 marks)

The points A, B and C have coordinates $(2, 1, -1)$, $(-2, 4, -2)$ and $(a, -5, 1)$ respectively, relative to the origin O, where $a \neq 10$.

Given $\overrightarrow{AB} \times \overrightarrow{AC} = (10-a)\underset{\sim}{j} + 3(10-a)\underset{\sim}{k}$ and the area of triangle ABC is $4\sqrt{10}$ square units, find the possible values of the constant a.

$$\text{Area of } \Delta = \frac{1}{2}\left|\overrightarrow{AB} \times \overrightarrow{AC}\right| = 4\sqrt{10}$$

$$= \frac{1}{2}\sqrt{[(10-a)^2 + 9(10-a)^2]} = 4\sqrt{10}$$

$$\sqrt{10(10-a)^2} = 8\sqrt{10} \checkmark$$

$$\sqrt{10}\,.\,|10-a| = 8\sqrt{10}$$

$$|10-a| = 8 \checkmark$$

[3]

$$a = 2 \text{ or } 18.$$ ✓ (both)

Question 3 (5 marks)

Solve the following system of equations by first forming an augmented matrix. Show each row operation beside the matrix.

$$x + y + z = 1$$

$$2x + y - z = -3$$

$$3x + 2y + z = 1$$

[5]

$$\left[\begin{array}{ccc|c} 1 & 1 & 1 & 1 \\ 2 & 1 & -1 & -3 \\ 3 & 2 & 1 & 1 \end{array}\right]$$

$$\left[\begin{array}{ccc|c} 1 & 1 & 1 & 1 \\ 0 & -1 & -3 & -5 \\ 0 & -1 & -2 & -2 \end{array}\right] \begin{array}{l} \\ R_2' = R_2 - 2R_1 \\ R_3' = R_3 - 3R_1 \end{array}$$

$$\left[\begin{array}{ccc|c} 1 & 1 & 1 & 1 \\ 0 & 1 & 3 & 5 \\ 0 & -1 & -2 & -2 \end{array}\right] \begin{array}{l} \\ R_2' = -R_2 \\ \\ \end{array}$$

$$\left[\begin{array}{ccc|c} 1 & 1 & 1 & 1 \\ 0 & 1 & 3 & 5 \\ 0 & 0 & 1 & 3 \end{array}\right] R_3' = R_3 + R_2$$

✓✓✓

$z = 3$.

$y + 9 = 5 \Rightarrow y = -4$

$x - 4 + 3 = 1 \Rightarrow x = 2$

$x = 2, \ y = -4, \ z = 3$ ✓✓

Question 4 (11 marks)

Referred to an origin O, the points A, B, C and D have coordinates $\overset{A}{(1,1,0)}$, $\overset{B}{(3,2,5)}$, $C\ (0,-1,-4)$ and $\underset{D.}{(-2,-5,0)}$ respectively.

a) Find the vector equation of the plane Π passing through A, B and C.

$$\overrightarrow{AB} = \begin{bmatrix} 2 \\ 1 \\ 5 \end{bmatrix} \quad \overrightarrow{AC} = \begin{bmatrix} -1 \\ -2 \\ -4 \end{bmatrix}$$

$$\overrightarrow{AB} \times \overrightarrow{AC}$$

$$\begin{vmatrix} i & j & k \\ 2 & 1 & 5 \\ -1 & -2 & -4 \end{vmatrix}$$

$$= (-4+10)\underline{i} - (-8+5)\underline{j} + (-4+1)\underline{k}$$

$$= 6\underline{i} + 3\underline{j} - 3\underline{k} = 3(2\underline{i} + \underline{j} - \underline{k})$$

$$\underline{r}\cdot\underline{n} = \underline{a}\cdot\underline{n}$$

$$\underline{r}\cdot\begin{bmatrix} 2 \\ 1 \\ -1 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}\cdot\begin{bmatrix} 2 \\ 1 \\ -1 \end{bmatrix} = 2+1 = 3$$

$$\underline{r}\cdot\begin{bmatrix} 2 \\ 1 \\ -1 \end{bmatrix} = 3.$$

[5]

The line l passes through D and is perpendicular to Π.

b) State a vector equation of l.

$$l: \quad \underline{r} = \begin{bmatrix} -2 \\ -5 \\ 0 \end{bmatrix} + \lambda\begin{bmatrix} 2 \\ 1 \\ -1 \end{bmatrix}$$

[1]

The line l meets the plane Π at the point E.

c) Find the coordinates of E.

$$\begin{bmatrix} -2+2\lambda \\ -5+\lambda \\ -\lambda \end{bmatrix}\cdot\begin{bmatrix} 2 \\ 1 \\ -1 \end{bmatrix} = 3$$

$$-4+4\lambda-5+\lambda+\lambda = 3$$

$$6\lambda = 12$$

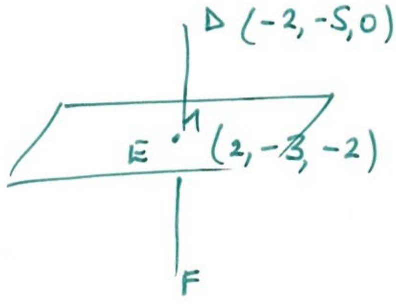
$$\lambda = 2.$$

$$E\ (2,-3,-2).$$

[3]

The point F is the reflection of D in Π.

d) Find the coordinates of F.



$$F\ (6,-1,-4).$$

[2]

Name: ____________________

Calculator assumed section **Suggested time: ~25 minutes** **/26**

Question 5 (8 marks)

The planes Π_1 and Π_2 are defined by the equations $\underset{\sim}{r}.\begin{pmatrix}2\\-1\\3\end{pmatrix} = 5$ and $x + 4y + z = -2$.

$\underset{\sim}{r}.\begin{bmatrix}1\\4\\1\end{bmatrix} = -2$

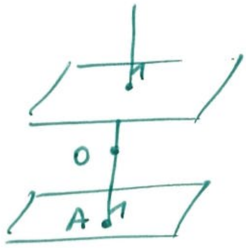
a) Find, to the nearest degree, the acute angle between Π_1 and Π_2.

$\text{angle}\left(\begin{bmatrix}2\\-1\\3\end{bmatrix}.\begin{bmatrix}1\\4\\1\end{bmatrix}\right) \Rightarrow \theta = 86^\circ$ nearest degree.

[2]

The point A has coordinates $(2, 1, -2)$.

b) Find the perpendicular distance between A and Π_1.



$\Pi_1 : \underset{\sim}{r} \cdot \frac{1}{\sqrt{14}}\begin{bmatrix}2\\-1\\3\end{bmatrix} = \frac{5}{\sqrt{14}}$

Plane $\parallel$ to Π_1 through A : $\underset{\sim}{r}.\begin{bmatrix}2\\-1\\3\end{bmatrix} = \begin{bmatrix}2\\1\\-2\end{bmatrix}.\begin{bmatrix}2\\-1\\3\end{bmatrix} = -3$

$\therefore \underset{\sim}{r}.\frac{1}{\sqrt{14}}\begin{bmatrix}2\\-1\\3\end{bmatrix} = \frac{-3}{\sqrt{14}}$

Distance $= \frac{8}{\sqrt{14}}$ or $\frac{4\sqrt{14}}{7}$

[3]

The plane Π_3 is perpendicular to Π_1 and Π_2 and the point with coordinates $(0, 4, -1)$ lies on Π_3.

c) Find the Cartesian equation of Π_3.

$\text{crossP}\left(\begin{bmatrix}2\\-1\\3\end{bmatrix} \times \begin{bmatrix}1\\4\\1\end{bmatrix}\right) = \begin{bmatrix}-13\\1\\9\end{bmatrix}$

$\Pi_3 : \underset{\sim}{r}.\begin{bmatrix}-13\\1\\9\end{bmatrix} = \begin{bmatrix}0\\4\\-1\end{bmatrix}.\begin{bmatrix}-13\\1\\9\end{bmatrix} = -5$

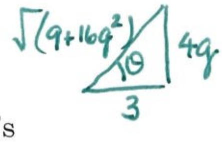
$\therefore \underset{\sim}{r}.\begin{bmatrix}-13\\1\\9\end{bmatrix} = -5$

[3]

$\Rightarrow -13x + y + 9z = -5$

$(\text{or } 13x - y - 9z = 5).$

Question 6 (6 marks)



A firework company is testing its new brand of firework. One of the company's employees lights the firework on a large area of horizontal ground and it takes off at a small angle to the vertical. After a flight lasting 8 seconds it lands at a distance of 24 metres from the launch point.

The employee models the firework as a particle and ignores air resistance and any loss of mass which the firework experiences.

Using this model, find for this flight of the firework:

a) the initial velocity vector

$\underline{i}$ cmpt: $8u_H = 24 \quad u_H = 3$ ✓

$\underline{j}$ cmpt: $8u_V - \frac{1}{2}g \cdot 64 = 0 \qquad u_V = 4g$ ✓ $\qquad \underline{u} = 3\underline{i} + 4g\underline{j}$ ✓

[3]

b) the initial speed, correct to 3 s.f.

$|3\underline{i} + 4g\underline{j}| = 39.3\ ms^{-1}$.

[1]

c) the maximum height attained

Max height $= \dfrac{|\underline{u}|^2 . \sin^2\theta}{2g.} = \dfrac{39.3^2}{2g} . \dfrac{16g^2}{(9+16g^2)}$ ✓ $= 78.34m.$ ✓

[2]

Question 7 (4 marks)

Determine the point of intersection of the two lines below. Explain your working.

$$\underline{r} = \begin{pmatrix} 8 \\ -1 \\ -8 \end{pmatrix} + \lambda \begin{pmatrix} 2 \\ 0 \\ -3 \end{pmatrix} \text{ and } \underline{r} = \begin{pmatrix} 0 \\ 1 \\ -3 \end{pmatrix} + \mu \begin{pmatrix} 1 \\ -1 \\ 2 \end{pmatrix}$$

For intersection

$$\begin{bmatrix} 8+2\lambda \\ -1 \\ -8-3\lambda \end{bmatrix} = \begin{bmatrix} \mu \\ 1-\mu \\ -3+2\mu \end{bmatrix}$$

$$\Rightarrow \left.\begin{aligned} 8+2\lambda &= \mu \\ -1 &= 1-\mu \\ -8-3\lambda &= -3+2\mu \end{aligned}\right\} \lambda, \mu \qquad \begin{aligned} \lambda &= -3 ✓ \\ \mu &= 2. ✓ \end{aligned}$$

check using $8+2\lambda = \mu \Rightarrow 8-6 = 2$ ✓. ✓

Point of intersection $\begin{bmatrix} 8 \\ -1 \\ -8 \end{bmatrix} - 3\begin{bmatrix} 2 \\ 0 \\ 3 \end{bmatrix} = \begin{bmatrix} 2 \\ -1 \\ 1 \end{bmatrix} \quad (2,-1,1).$ ✓

[4]

or $\begin{bmatrix} 0 \\ 1 \\ -3 \end{bmatrix} + 2\begin{bmatrix} 1 \\ -1 \\ 2 \end{bmatrix} = \begin{bmatrix} 2 \\ -1 \\ 1 \end{bmatrix}.$

Question 8 (8 marks)

The position vector of a particle at time t seconds is given by$\underline{r} = (t^3 - 2t)\underline{i} + (t^2 - 3t)\underline{j}$.

a) Find the speed of the particle at time $t = 3$ seconds.

$\underline{v} = \dot{r}(t) = (3t^2 - 2)\underline{i} + (2t - 3)\underline{j}$

speed $= |v(3)| = \sqrt{(25^2 + 3^2)} = 25.18\ ms^{-1}$

[3]

b) Does the particle ever come to a stop? If so, when and where does it stop? If not, explain why not.

Need $\underline{i}$ and $\underline{j}$ components of $v(t) = 0$.

$3t^2 - 2 = 0$ and $2t - 3 = 0$

$t = \sqrt{\frac{2}{3}}$ $\quad t = \frac{3}{2}$

t values not equal $\Rightarrow$ particle doesn't stop.

[3]

c) Find the position vector when the particle is moving parallel to the horizontal axis.

rate of change of $\underline{j}$ component $= 0$.

[2]

$\frac{d}{dt}(t^2 - 3t) = 0$

$2t - 3 = 0 \Rightarrow t = \frac{3}{2}$

$\underline{r}\left(\frac{3}{2}\right) = \frac{3}{8}\underline{i} - \frac{9}{4}\underline{j}$

(or $0.375\underline{i} - 2.25\underline{j}$).